数理解析研究所講究録
432巻 1981年 63-73

Uniform ultimate boundedness of limiting equations in retarded system

Junji KATO
(Mathematical Institute, Tohoku University)

The relationship between the stability (including the boundedness) of a system and that of the limiting equations has been discussed by many authors. For a part of references, see [1]. Recently we have added some remarks including several illustrative examples which show that substantial conditions can not be dropped in order that the stability property is inherited to the limiting equations [1].

In this report, we shall show that the most of the results in [1] can be extended to functional differential equations though the uniform ultimate boundedness no longer implies the uniform boundedness even for autonomous systems, see [2].

Let Y and Z be metric spaces, and denote by $C(Y, Z)$ the space of Z-valued continuous functions defined on Y. A sequence $\{f^k\}_k$ in $C(Y, R^n)$ is said to be <u>c-uniformly convergent</u> on Y if it is convergent uniformly on any compact set in Y. Clearly a c-uniformly convergent sequence has a unique limit which belongs to $C(Y, R^n)$. Assume that $Y = I \times X$ is a product space with $I = [0, \infty)$. A function $f(t, \phi) \in C(I \times X, R^n)$ is said to be <u>(positively) precompact</u> if for any sequence

$\{t_k\}$ in I the sequence $\{f(t+t_k\,,\phi)\}_k$ contains a c-uniformly convergent subsequence. We denote by $\Omega(f)$ the set of all limit functions of c-uniformly convergent sequences $\{f(t+t_k\,,\phi)\}_k$ on $I \times X$ for sequences $\{t_k\}$ such that $t_k \to \infty$, while we set $T(f) = \{f(t+\tau\,,\phi) : \tau \in I\}$ and $H(f)$ denotes the closure of $T(f)$ in $C(I \times X\,, R^n)$ under the compact-open topology.

Then we have the following lemma.

Lemma 1. (i) $H(f) = T(f) \cup \Omega(f)$.

(ii) If $f \in C(I \times X\,, R^n)$ is precompact, then it is bounded and uniformly continuous on $I \times K$ for any compact set $K \subset X$, and the converse holds if X is separable.

(iii) For any $g \in H(f)$ we have $H(g) \subset H(f)$. Moreover, if X is separable and f is precompact, then $H(g) \subset \Omega(f)$ for any $g \in \Omega(f)$ and, hence, we have $H(f) = \Omega(f)$ under the condition

(1) $$f \in \Omega(g) \quad \text{for any} \quad g \in \Omega(f)\,.$$

Proof. The assertion (i) and the first parts of (ii) and (iii) are obvious. We shall prove the converse part in (ii). Let $\{(\tau_\ell\,,\phi^\ell)\}_\ell$ be a countable dense set in $I \times X$, and let $\{t_k\}$ be any sequence in I. By the assumption $\{f(\tau_\ell + t_k\,,\phi^\ell)\}_k$ contains a convergent subsequence for any fixed ℓ, and hence by the standard argument, we can select a subsequence of $\{t_k\}$, which is again denoted by $\{t_k\}$, so that $\{f(\tau_\ell + t_k\,,\phi^\ell)\}_k$ is convergent for every ℓ. Let K be any compact set in

$I \times X$, and let $A_m \subset \{(\tau_\ell, \phi^\ell)\}_\ell$ be a minimal finite $1/m$-net of K, where an ε-net A is said to be <u>minimal</u> if any proper subset of A is no longer ε-net. Put $K^* = K \cup [\cup_{m=1}^\infty A_m]$. Clearly K^* is compact since the derived set is contained in K. We shall show that $\{f(t_k + t, \phi)\}_k$ converges uniformly on K, which implies that it is c-uniformly convergent on $I \times X$. Since $f(t, \phi)$ is uniformly continuous on $I \times \{\phi : (t, \phi) \in K^*$ for a $t \in I\}$, there is a $\delta(\varepsilon) > 0$ for any $\varepsilon > 0$ such that $|f(\tau + t, \phi) - f(\tau + s, \psi)| < \varepsilon$ for any $\tau \in I$ if (s, ψ) belongs to a $\delta(\varepsilon)$-neighborhood of (t, ϕ) in K^*. On the other hand, for any integer m and $\varepsilon > 0$ we can find an $N_m(\varepsilon)$ so that $|f(t_k + t, \phi) - f(t_j + t, \phi)| < \varepsilon$ if $(t, \phi) \in A_m$ and $k, j \geqq N_m(\varepsilon)$. For a given $\varepsilon > 0$ choose an integer $m(\varepsilon)$ so that $m(\varepsilon)\delta(\varepsilon/3) > 1$, and set $N(\varepsilon) = N_{m(\varepsilon)}(\varepsilon/3)$. Then, for any $(t, \phi) \in K$ there is an $(s, \psi) \in A_{m(\varepsilon)}$ which belongs to a $\delta(\varepsilon/3)$-neighborhood of (t, ϕ). Therefore, if $k, j \geqq N(\varepsilon)$, then $|f(t_k + t, \phi) - f(t_j + t, \phi)| \leqq |f(t_k + t, \phi) - f(t_k + s, \psi)| + |f(t_k + s, \psi) - f(t_j + s, \psi)| + |f(t_j + s, \psi) - f(t_j + t, \phi)| < \varepsilon$. This completes the proof. This is actually a consequence of the Ascoli-Arzela theorem.

Now we shall prove that $\Omega(f)$ is closed if X is separable and if f is precompact, which assures the second part of (iii) since $T(g) \subset \Omega(f)$ for any $g \in \Omega(f)$. Let $\{g^k\}_k$ be a sequence in $\Omega(f)$ which converges to a g c-uniformly on $I \times X$. Since $g^k \in \Omega(f)$, there is a divergent sequence $\{t_{kj}\}_j$ for which $\{f(t_{kj} + t, \phi)\}_j$ converges to $g^k(t, \phi)$ c-uniformly on $I \times X$. It is not difficult to see that by choosing a suitable j_k for each k and setting $s_k = t_{kj_k}$ we can claim that $\{f(s_k + \tau_\ell , \phi^\ell)\}_k$ converges to $g(\tau_\ell , \phi^\ell)$ for every ℓ, where $\{(\tau_\ell , \phi^\ell)\}_\ell$ is a given countable dense set of $I \times X$ under the separability condition of X. Now the same argument as in the proof of (ii) asserts that $\{f(s_k + t , \phi)\}_k$ converges to $g(t , \phi)$ c-uniformly on $I \times X$, that is, $g \in \Omega(f)$.

Remark. Let K be a compact set in a separable metric space (Y , d_Y). There is no doubt for the existence of a minimal finite ε-net A_ε of K, where A_ε is said to be an ε-net of K if $\cup\{U_\varepsilon(x) : x \in A_\varepsilon\} \supset K$ for

$$(2) \qquad U_\varepsilon(x) = \{y \in Y : d_Y(x , y) < \varepsilon\}$$

or

$$(3) \qquad U_\varepsilon(x) = \{y \in Y : d_Y(x , y) \leqq \varepsilon\} .$$

In the case of (2) it should be noted that every minimal ε-net A_ε of K is always finite. In fact, if $\{x^k\}_k \subset A_\varepsilon$ be infinite, then from the minimality there is a $y^k \in K$ such that $y^k \in U_\varepsilon(x^k) \setminus U_\varepsilon(x)$ for all $x \in A_\varepsilon \setminus \{x^k\}$. The compactness of K allows us to assume that y^k converges to a $y \in K$. Choose $x \in A_\varepsilon$ so that $y \in U_\varepsilon(x)$. Then, we have $y^k \in U_\varepsilon(x)$ for all large k, which yields a contradiction.

However, in the case of (3), a minimal ε-net is not necessarily finite nor even (relatively) compact. For example, choose an $x^k \in C([0 , 1] , R^1)$ defined by

$$x^k(t) = \begin{cases} 1 & t = \frac{1}{k} \\ 0 & |t - \frac{1}{k}| \geqq \frac{1}{2k^2} \\ \text{linear} & \text{otherwise}, \end{cases}$$

and set $K = \{\frac{1}{k}(x^k - x^1)\}_k$ and $A_\varepsilon = \{(\varepsilon + \frac{1}{k})x^k - \frac{1}{k}x^1\}_k$. Clearly, K is a compact set in the separable Banach space $C([0,1], R^1)$, and A_ε is a minimal ε-net of K but not relatively compact.

We shall consider the functional differential equation

$$\dot{x}(t) = f(t, x_t) \tag{4}$$

defined on $I \times X$, where $x_t(s) = x(t+s)$ for $s \leqq 0$. Here and henceforth X is a space of R^n-valued functions defined on $(-\infty, 0]$ with a pseudometric $d(\cdot,\cdot)$ satisfying the following properties [3]: The zero function θ always belongs to X.

(H_0) The corresponding metric space X/d is complete.

(H_1) The mapping : $\phi \longrightarrow \phi(0)$ belongs to $C(X, R^n)$.

(H_2) $x_t \in X$ if $(t, x) \in [0, a] \times X_a$, and the mapping $(t, x) \longrightarrow x_t$ belongs to $C([0, a] \times X_a, X)$ for any $a > 0$, where X_a is the set of the R^n-valued functions x defined on an interval containing $(-\infty, a]$ such that $x_0 \in X$ and $x \in C([0, a], R^n)$ with the pseudometric

$$d_a(x, y) = \max\{d(x_0, y_0), \sup_{0 \leqq t \leqq a}|x(t) - y(t)|\}.$$

Throughout this paper we assume that f is continuous on $I \times X$ and positively precompact. The continuity of f allows us to consider that f is defined on the metric space X/d, which we shall identify with X. A system

$$(5) \qquad \dot{x}(t) = g(t, x_t)$$

is said to be a limiting equation of (4) when $g \in \Omega(f)$. We say that the system (4) is regular if the solutions of every limiting equation of (4) are unique for the initial valued problem.

The following lemma is a special case of the theorem in [3] which is a version of the well-known Kamke's theorem.

Lemma 2. Suppose that $\{f(t+t_k, \phi)\}_k$ converges to a $g(t, \phi)$ c-uniformly on $I \times X$, and let $x^k(t)$ be a non-continuable solution of (4) such that $x^k_{t_k}$ converges to a $\xi \in X$. Then, the sequence $\{x^k(t+t_k)\}_k$ contains a subsequence which converges to a solution $x(t)$ of (5) through ξ at $t=0$ c-uniformly on a domain of $x(t)$. Furthermore, if $x(t)$ is the unique solution of (5) through ξ at $t=0$, then $\{x^k(t+t_k)\}_k$ itself must converge to $x(t)$.

Applying this lemma, we can extend Theorem 1 in [1] without any effort. Namely we have the following.

Theorem 1. Suppose that the solutions of every limiting equation of (4) are continuable up to $t = \infty$. If the solutions of (4) are compact-uniformly ultimately bounded, then they are compact-uniformly bounded.

In the above, the solutions of (4) are said to be compact-uniformly ultimately bounded if there are a constant $B > 0$ and a number $\sigma(\Gamma) > 0$ depending on every compact set $\Gamma \subset X$ such that any solution $x(t)$ of (4) with $x_\tau \in \Gamma$ for a $\tau \in I$ satisfies $|x(t)| \leqq B$ for all $t \geqq \tau + \sigma(\Gamma)$. This concept corresponds to the compact dissipative when (4) produces a dynamical system, refer to [4]. The definition of the compact-uniform boundedness will be given in a same manner. If in the above Γ ranges over bounded sets of X, then we shall omit the prefix "compact" in the definitions. Note that if X is a locally compact normed space, there is no difference whether the prefix "compact" is omitted or not. However, as was shown in [2] we can not omit this prefix in Theorem 1.

The space X is said to have a fading memory, if X satisfies the conditions : For any $a > 0$,

(H_3) $d(x_t , y_t) \leqq \beta(\alpha)$ if $t \in [0 , a]$ and $d_a(x , y) \leqq \alpha$,

(H_4) $d(x_t , y_t) \leqq \varepsilon$ for all $t \in [\tau + \omega(\varepsilon , \alpha) , a]$ if $d_a(x , y) \leqq \alpha$ and $|x(t) - y(t)| < \delta(\varepsilon)$ for $t \in [\tau , a]$,

where $\beta(\alpha)$, $\delta(\varepsilon)$ and $\omega(\varepsilon , \alpha)$ are positive numbers depending on their indicated arguments.

Lemma 3. Suppose that X has a fading memory. Then, the sequence $\{x^k_{t_k}\}$ contains a convergent subsequence if $x^k \in S(\alpha , L) = \bigcap_{a>0} \{x \in X_a : d_a(x , \theta) \leqq \alpha , |x(t) - x(s)| \leqq L|t - s|$ on $[0 ,$

a]} and if $t_k \to \infty$, where α, L are positive constants and θ is the zero function on $(-\infty, \infty)$.

Proof. Take a sequence $\{x^k_{t_k}\}$ for $x^k \in S(\alpha, L)$ with $t_k \to \infty$, and set $y^k(t) = x^k(t + t_k - a)$ for an $a > 0$. Then, we have $x^k_{t_k} = y^k_a$ and $y^k \in S(\beta(\alpha), L)$ if $t_k \geqq a$, where $\beta(\alpha)$ is the one in (H_3). Here, we may assume that $y^k(t)$ converges to a $y(t)$ c-uniformly on $[0, \infty)$. For any $\varepsilon > 0$, choose N so large that $t_k > a = \omega(\varepsilon, \beta(\alpha))$ and $|y^k(t) - y(t)| \leqq \delta(\varepsilon)$ on $[0, a]$ if $k \geqq N$. Hence, the condition (H_4) assures that $d(y^k_a, y_a) \leqq \varepsilon$ if $k \geqq N$, that is, $x^k_{t_k}$ converges to y_a.

It is easily seen that every result in [1] can be extended to the equation (4) in a same manner as for Theorem 1. Under additional conditions we have the following theorem which corresponds to Theorem 2 in [1].

Theorem 2. In addition to the assumption that X is separable and has a fading memory, suppose that $f(t, \phi)$ satisfies the condition (1) and is bounded on $I \times \Gamma$ for every bounded set $\Gamma \subset X$ and that the solutions of every limiting equation of (4) are uniformly bounded. Then, the solutions of (5) are uniformly ultimately bounded for every $g \in H(f)$ if so are the solutions of (4).

Proof. Suppose that there is a $g \in H(f)$ for which the solutions of (5) are not uniformly ultimately bounded. Then, there exist a constant $\alpha > 0$, sequences $\{\tau_k\}$, $\{t_k\}$ and $\{x^k(t)\}$, solutions of (5), such that $\tau_k \geqq 0$, $t_k - \tau_k$ $(= 2s_k) \geqq k$, $d(x^k_{\tau_k}, \theta) \leqq \alpha$ and $|x^k(t)| \geqq B + 1$ on $[\tau_k, t_k]$, because otherwise we will have $|x(t)| \leqq \gamma(B+1)$ for all $t \geqq \tau + \sigma$ and some $\sigma > 0$ if $d(x_\tau, \theta) \leqq \alpha$, where B is a bound for the uniform ultimate boundedness of (4) and $\gamma(\alpha) \geqq \alpha$ is the number which is associated with the definition of the uniform boundedness of (5). Here, note that $g \in H(f) = \Omega(f)$ by Lemma 1. Since $|x^k(t)| \leqq \gamma(\alpha)$ for all $t \geqq \tau_k$, the sequence y^k defined by $y^k(t) = x^k(t + \tau_k)$ belongs to $S(\beta(\gamma(\alpha)), L(\alpha))$, where $L(\alpha)$ is a bound for $|f(t, \phi)|$ on $I \times \{\phi : d(\phi, \theta) \leqq \beta(\gamma(\alpha))\}$ and $\beta(\alpha)$ is the one in (H_3). Thus, by Lemma 3 $\{y^k_{s_k}\}_k$ contains a convergent subsequence, namely, we may assume that $\{y^k_{s_k}\}_k$ is convergent to a $\xi \in X$. On the other hand, by applying Lemma 2 we may also assume that $x^k(t + \tau_k + s_k) = y^k(t + s_k)$ converges to an $x(t)$ c-uniformly on $[0, \infty)$ and $\{g(t + \tau_k + s_k, \phi)\}_k$ converges to an $h(t, \phi)$ c-uniformly on $I \times X$, where $x(t)$ is a solution of

$$\dot{x}(t) = h(t, x_t)$$

through ξ at $t = 0$. Clearly $|x^k(t + \tau_k + s_k)| \geqq B + 1$ on $[0, t_k - \tau_k - s_k] = [0, s_k]$ implies that $|x(t)| \geqq B + 1$ on $[0, \infty)$. Since $h \in \Omega(g) \subset H(f) = \Omega(f)$ by Lemma 1, we have $f \in \Omega(h)$ by (1) and there is a sequence $\{u_k\}$ for which $\{h(t + u_k,$

$\phi)\}_k$ converges to $f(t, \phi)$ c-uniformly on $I \times X$. Thus, applying Lemma 2 again we can see that $\{x(t+u_k)\}_k$ converges to a solution $y(t)$ of (4), and we have $|y(t)| \geqq B+1$ on $[0, \infty)$ a contradiction.

When the system (4) is regular, we can omit the additional conditions and the following statements given in [1] can be reproduced with the same proofs.

Theorem 3. Assume that the system (4) is regular and that the solutions of (5) are continuable up to $t = \infty$ for every $g \in \Omega(f)$. Then, if the solutions of (4) are uniformly ultimately bounded with a bound B, then so are the solutions of (5) for every $g \in H(f)$.

Theorem 4. If the system (4) is regular and if its solutions are uniformly bounded, then the solutions of (5) are uniformly bounded with the same pair $(\alpha, \gamma(\alpha))$ for every $g \in H(f)$.

References

[1] J. Kato and T. Yoshizawa, Remarks on global properties in limiting equations, to appear in Funkcialaj Ekvacioj.
[2] J. Kato, An autonomous system whose solutions are uniform-

ly ultimately bounded but not uniformly bounded, Tohoku Math. J. 32 (1980), 499-504.

[3] J. Kato, Kamke theorem in functional differential equations, to appear.

[4] J. K. Hale, Theory of Functional Differential Equations, Springer-Verlag, Berlin-Heidelberg-New York, 1977.